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Tracing the String:

BMN Correspondence at Finite J^2/N

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Abstract

Employing the string bit formalism of hep-th/0209215, we identify the basis transformation that relates BMN operators in $\mathcal{N} = 4$ gauge theory to string states in the dual string field theory at finite $g_2 = J^2/N$. In this basis, the supercharge truncates at linear order in g_2, and the mixing amplitude between 1 and 2-string states precisely matches with the (corrected) answer of hep-th/0206073 for the 3-string amplitude in light-cone string field theory. Supersymmetry then predicts the order g_2^2 contact term in the string bit Hamiltonian. The resulting leading order mass renormalization of string states precisely matches with the recently computed shift in conformal dimension of BMN operators in the gauge theory.

Introduction and Philosophy

The BMN correspondence [1] equates type IIB string theory on a plane wave background with a certain limit of $\mathcal{N}=4$ gauge theory at large R-charge J, where N is taken to infinity while the quantities

$$\lambda' = \frac{g_{\rm YM}^2 N}{J^2}, \qquad g_2 = \frac{J^2}{N} \tag{1}$$

are held fixed. The proposal is based on a natural identification between the basis of string theory states and the basis of gauge theory operators, and between the light-cone string Hamiltonian P^- and the generator Δ of conformal transformation in the gauge theory via[1]

$$\frac{2}{\mu} P^- = \Delta - J\,. \tag{2}$$

BMN argued, and it was subsequently confirmed to all orders in λ' [2, 3], that this identification holds at the level of free string theory ($g_2 = 0$).

This beautiful proposal equates two operators which act on completely different spaces: the light-cone Hamiltonian P^- acts on the Hilbert space of string field theory, and allows for the splitting and joining of strings, while $H \equiv \Delta - J$ acts on the operators of the field theory, and in general mixes single-trace operators with double- and higher-trace operators. Light-cone string field theory in the plane wave background has been constructed in [4, 5]. On the field theory side, a number of impressive papers [7–14] have pushed the calculations to higher order in g_2 with the aim of showing that (2) continues to hold, thereby providing an equality between a perturbative, interacting string theory and perturbative $\mathcal{N}=4$ gauge theory. It is clear, however, that at finite g_2 the natural identification between single string states and single trace operators breaks down. For example, 1-string states are orthogonal to 2-string states for all g_2, but single-trace operators and double-trace operators are not. This raises the question how to formulate the BMN correspondence in the interacting string theory.

In order to prove that two operators in (2) are equal, it is sufficient to prove that they have the same eigenvalues. If they do, then there is guaranteed to exist a unitary transformation between the spaces on which the two operators act. A basis independent formulation of the BMN correspondence, therefore, is that the interacting string field theory Hamiltonian $\frac{2}{\mu}P^-$ and the gauge theory operator H must have the same eigenvalues.

While this is the minimum that we are allowed to expect from the BMN correspondence, we can hope to do better. Light-cone string field theory, as formulated in [4–6], comes with a natural choice of basis: this string basis (of single and multiple strings) is neither the BMN basis (of single and multiple traces) nor the basis of eigenstates of the light-cone Hamiltonian. But how do we identify the string basis in the gauge theory?

[1]The parameter μ can be introduced by performing a boost and serves merely as a bookkeeping device.

One guess for the string basis was made in [9, 13], where it was argued that matrix elements of P^- between 1- and 2-string states should be equated with the coefficient of the three-point function of the corresponding BMN operators, multiplied by the difference in conformal dimension between the incoming and outgoing states. This proposal appeared to be supported by the subsequent string field theory calculation done in [5] (see also [18–25]). It turns out, however, that the final step of the calculation in [5] suffered from a minus sign error (which we will correct below), which renders the alleged confirmation of this proposal invalid.

In this paper we propose a new, specific form for the transformation between the BMN basis and the string basis, valid to all orders in g_2. This basis transformation is trivial to write down, and has the pleasing feature that it does not depend on the conformal dimensions of the operators. In fact, our choice of transformation was already identified as a natural choice in [16], where it was shown that all computed amplitudes in gauge theory are reproduced via a relatively simple string bit formalism. While most calculations in [16] were done in the BMN basis, it was pointed out that there exists a basis choice with the properties that (i) the inner product is diagonal, and (ii) the matrix elements of the supersymmetry generators Q are at most linear in g_2 (i.e. Q leads to only a single string splitting or joining). Here we will show that, when evaluated in this new basis, the matrix elements of the string bit Hamiltonian, which via the results of [16] may be identified with the gauge theory operator on the right-hand side of (2), agree precisely with the corrected answer of [5] for the matrix elements of the continuum string field theory Hamiltonian P^- appearing on the left-hand side!

The precise match between the three point functions means that, by combining the two formalisms, we can start filling in some important questions left open in [4] and [16]. A major technical obstacle in continuum light-cone string field theory is that higher order contact terms are needed for closure of the supersymmetry algebra, and that their value (at order g_2^2) affects the leading order shift in the eigenvalues of P^-. However, these contact terms are difficult to compute [17]. The supersymmetry algebra of the bit string theory, on the other hand, is known to all orders in g_2 but only to linear order in the fermions. It appears to be a fruitful strategy, therefore, to make use of the discretized theory to fix the order g_2^2 contact terms of the continuum theory, while the known non-linear fermionic form of the continuum interaction vertex may be of direct help in deriving the complete supersymmetry generators in the string bit formalism.

Identification of the String Basis in Gauge Theory

$\mathcal{N} = 4$ gauge theory in the BMN limit comes with a natural choice of basis, which coincides with the natural string basis when $g_2 = 0$: an n string state corresponds to a product of n single trace BMN operators. We call this basis the BMN basis, denoted by $|\psi_n\rangle$. At non-zero g_2, the inner product (defined as the overlap as computed in the free gauge theory) becomes non-diagonal in this basis. The explicit form of the inner product is conveniently expressed in terms of the string bit language of [15, 16] as

$$\langle\psi_m|\psi_n\rangle_{g_2} = \langle\psi_m|\, e^{g_2\Sigma}\,|\psi_n\rangle_0\,, \qquad\qquad \Sigma = \frac{1}{J^2}\sum_{m<n}\Sigma_{mn} \tag{3}$$

where Σ_{mn} is the operator which interchanges the string bits via he simple permutation (mn). As explained in [15, 16], when acting on a BMN state $|\psi\rangle$ with n strings, Σ effectuates a single string splitting or joining.

This meaning of Σ in the gauge theory language can be made concrete as follows. Consider a long BMN string in its ground state. We can write the corresponding operator as

$$\mathcal{O}_J(\gamma) = \mathrm{Tr}(Z^J) = \sum_{\substack{i_1\ldots i_J\\ \bar{i}_1\ldots\bar{i}_j}} Z_{i_1\bar{i}_1}\, Z_{i_2\bar{i}_2}\ldots Z_{i_J\bar{i}_J}\; \delta^{\bar{i}_1 i_{\gamma(1)}}\,\delta^{\bar{i}_2 i_{\gamma(2)}}\ldots\delta^{\bar{i}_J i_{\gamma(J)}} \tag{4}$$

with $\gamma = (12\ldots J)$ the cyclic permutation of J elements. The action of $\Sigma_{J_1 J}$, which implements the simple permutation $(J_1 J)$, is now defined as

$$\Sigma_{J_1 J}\,\mathcal{O}_J(\gamma) = \mathcal{O}_J(\gamma\circ(J_1 J)) \tag{5}$$

Since $\gamma\circ(J_1 J) = (1\ldots J_1{-}1\; J)(J_1\ldots J-1)$ we have that

$$\Sigma_{J_1 J}\,\mathcal{O}_J(\gamma) = \mathrm{Tr}(Z^{J_1})\mathrm{Tr}(Z^{J-J_1}), \tag{6}$$

showing that the simple permutation $\Sigma_{J_1 J}$ indeed induces a single splitting of a single trace into a double trace operator. It is easy to generalize this result to other operators, to show that Σ can either split a string or join two strings.

The identification of (3) with the inner product of the free gauge theory was motivated in [16] and explicitly verified for string ground states to all order in g_2 and for two-impurity states to order g_2^2.

States with different number of strings are therefore no longer orthogonal relative to (3). In the string field theory basis $|\tilde{\psi}_n\rangle$, on the other hand, the inner product should be diagonal for all g_2. The simplest basis transformation that achieves this goal is

$$|\widetilde{\psi}_n\rangle = (e^{g_2\Sigma/2})_{nm}|\psi_m\rangle\,. \tag{7}$$

This is not the most general diagonalization, however, since we still have the freedom to redefine the new basis $|\widetilde{\psi}_m\rangle$ via an arbitrary unitary transformation [12]. The above redefinition (7), however, has the attractive feature that it is purely combinatoric and does not depend on the dynamics of the gauge theory. Furthermore, as we will see shortly, it has the desirable property that the (linearized) supersymmetry generators and light-cone Hamiltonian acquire a simple form in the new basis. We emphasize that the only way to check the proposal (7) for identifying the string field theory basis in the gauge theory is by comparing matrix elements of H calculated in the $|\widetilde{\psi}_n\rangle$ basis to those of $\frac{2}{\mu}P^-$ in light-cone string field theory. We show below that the proposal (7) passes this test.

In the following, we will study the consequences of this basis transformation for the specific class of two-impurity BMN states investigated in [7, 9, 11–13]. We will denote by $|1,p\rangle$ the normalized state corresponding to the single trace operator $\sum_l e^{2\pi i p l/J}\mathrm{Tr}(\phi Z^l \psi Z^{J-l})$, while $|2,k,y\rangle$ and $|2,y\rangle$ will denote the normalized states corresponding to the double trace operators $\sum_l e^{2\pi i k l/J_1}\mathrm{Tr}(\phi Z^l \psi Z^{J_1-l})\mathrm{Tr}(Z^{J-J_1})$ and $\mathrm{Tr}(\phi Z^{J_1})\mathrm{Tr}(\psi Z^{J-J_1})$ respectively, where $y = J_1/J$. The action of Σ on $|1,p\rangle$ reads

$$\Sigma\,|1,p\rangle = \sum_{k,y} C_{pky}\,|2,k,y\rangle \;+\; \sum_{y} C_{py}|2,y\rangle\,, \tag{8}$$

with

$$C_{pky} = \sqrt{\frac{1-y}{Jy}}\,\frac{\sin^2(\pi p y)}{\pi^2(p-k/y)^2}\,, \qquad C_{py} = -\frac{\sin^2(\pi p y)}{\sqrt{J}\pi^2 p^2}\,. \tag{9}$$

Via (7) we now introduce the corresponding two-impurity states in the string basis, which we will denote by $|\tilde{1},p,y\rangle$, $|\tilde{2},k,y\rangle$ and $|\tilde{2},y\rangle$, respectively. By construction, these form an orthonormal basis at finite g_2.

Interactions in the String Basis

In this section we obtain the matrix elements of the right-hand side of (2) in the string basis proposed in the previous section. For this we will employ the string bit model of [16], but by virtue of the established correspondence with the gauge theory amplitudes of [11, 13], the following calculation can also be viewed as a direct calculation within the gauge theory.

It was shown in [16] that the linearized (in the fermions) interacting supercharges in the string bit model can be written in the string basis as

$$Q = Q_0 + \frac{g_2}{2}[\widehat{Q}_0, \Sigma]\,, \qquad\qquad \widehat{Q}_0 = Q_0^{<} - Q_0^{>}\,, \tag{10}$$

where $Q_0 = Q_0^{<} + Q_0^{>}$ is the free supercharge of the bit string theory and the superscripts indicate the projection onto the term with fermionic creation ($<$) or annihilation ($>$) operators